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Twenty-One Algebraic Normal Forms of Citrabhānu

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In this article we examine Citrabhānu's (fl. 1530) theory of algebraic normal forms up to the third degree as handed down to us by his pupil Śaṅkara and make a survey of their history in India. © 1998 Academic Press

この論考で我々は，シャンカラによって伝えられたチトラバーヌ（1530 年ころ活躍）の三次までの標準型方程式の理論を考察し，あわせてインドにおけるその歴史を調査する． © 1998 Academic Press

本文考察了由珊加拉所相伝的斉特拉巴努（活躍于 1530 年頃）的自一次至三次標準型方程式，並探討了在印度的那些方程式歴史． © 1998 Academic Press

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1. INTRODUCTION

Kerala of South India was productive of mathematicians and astronomers especially during the 14th to the 17th centuries A.D. It is in this period that Mādhava (fl. A.D. 1380/1420) and his successors discovered power series expansions of π and of trigonometric functions such as sine, cosine, arctangent, and versed sine. Although Mādhava's astronomical manuals are extant, much of his mathematical achievements can be known only through the works of his successors including, among other scholars, Nīlakaṇṭha (born 1444 and died after 1542) and his student, Śaṅkara Vāriyar (fl.ca.1556).

Citrabhānu, too, was a student of Nīlakaṇṭha and another teacher of Śaṅkara. He wrote an astronomical manual entitled *Karaṇāmṛta* in A.D. 1530. He is also known to have been the author of a commentary on the first three cantos of a literary work (*kāvya*), *Kirātārjunīya*, of Bhāravi (6th century) [3,A3:47a; 3,A4:93a; 3,A5:109b; 4,153–154; 7,57].

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Another source of information about his works is Śaṅkara's commentary, *Kriyākramakarī*, on the famous book on arithmetic and mensuration, *Līlāvatī* (before A.D. 1150), of Bhāskara II (born 1114 and died after 1183). While commenting on Bhāskara II's rules for two types of algebraic normal forms (see Section 3 below), Śaṅkara mentions a small treatise written by Citrabhānu and explains it "according to his (i.e., Citrabhānu's) instruction" (see his introductory remark translated in Section 2.2 below). The work, which seems to have been named "Solutions to the Twenty-One Problems,"[1] systematically treats 21 algebraic normal forms, that is, pairs of simultaneous equations in two unknown quantities, up to the third degree (see Section 2.1 below). Some of those normal forms had been treated by Indian mathematicians since at least the time of Āryabhaṭa I (born A.D. 476), but their treatments were unsystematic and restricted only up to the second degree (see Section 3 below). In Śaṅkara's text, Citrabhānu's rules are accompanied by illustrative examples and by proofs of the rules. Some of these proofs are geometrical; geometric treatment of algebraic problems is one of the characteristic features of the mathematics of the Kerala school.

The mathematics of Citrabhānu as handed down to us by Śaṅkara was studied first by T. A. Sarasvati [5, 230–234] and then by Gupta [1]. The study of Sarasvati was restricted to those seven rules which are accompanied by geometric proofs, since her chief concern was "geometrical algebra." Sarasvati gave modern, algebraic expressions of the seven rules and free English translations of the accompanying geometric proofs. Gupta, on the other hand, gave the entire scheme of Citrabhānu's 21 rules and discussed several typical cases. He cited two rules in Sanskrit, gave their English translations, and also referred to one example given in the text, but did not concern himself with the proofs in the text.

The present paper aims at giving a fuller exposition of Citrabhānu's treatment of the 21 algebraic normal forms. We will give all the rules in Sanskrit, their literal English translations and modern, algebraic expressions, the numerical examples in the text, literal English translations of the geometric proofs, and modern expositions of the algebraic proofs.[2]

Śaṅkara calls Citrabhānu "a leading scholar who knows the reasoning of mathematics and of the sphere (i.e., astronomy)." We hope our study will add a bit to our knowledge about this highly esteemed, but little known, mathematician.

[1] An anonymous Malayalam manuscript (GOML, Madras, No. Mal. D. 217) entitled *Ekavimśatipraśnottara* ("Solutions to the Twenty-One Problems"), whose contents are similar to those of the treatise under consideration, exists according to K. V. Sarma, the editor of the *Kriyākramakarī* (see his footnotes in [8, 121, 126]). It is most probable that the manuscript contains Citrabhānu's own work, but our study is, like those of our predecessors, based on Śaṅkara's presentation of Citrabhānu's rules, examples, and proofs in Sanskrit, since we are not in a position to read the Malayalam manuscript. Elsewhere [7, 14] Sarma lists an anonymous Sanskrit manuscript (in Malayalam characters) having a similar title (*Ekaviṃśatipraśnakrama*, No. 541–D of the curator's office collection, Mss. Library, Kerala University), which contains "an interesting summary of the general mathematical processes" [6, 36].

[2] We have amended the text in several places. See Appendix, Section 1.

2. NORMAL FORMS OF CITRABHĀNU

2.1. Synopsis

The rules of Citrabhānu and their correlations may be summarized as follows. Suppose x and y are two unknown numbers, and put $x + y = a$, $x - y = b$, $xy = c$, $x^2 + y^2 = d$, $x^2 - y^2 = e$, $x^3 + y^3 = f$, and $x^3 - y^3 = g$. Let any two of the seven values (a, b, c, d, e, f, g) be given (the number of possible combinations is ${}_7C_2 = (7 \times 6)/2 = 21$). Then Rule 1 and its variations, Rules 16 and 21, directly produce x and y, x^2 and y^2, and x^3 and y^3, respectively, from a and b, d and e, and f and g. The other rules produce any one of a, b, c, etc. (together with x or y in the cases of Rules 17 and 19) from other combinations, and one can obtain a and b, and hence x and y, by the application of other rules as indicated in the square brackets.

It should be noted here that 7 out of the 21 rules are not universal. Rules 6, 10, 15, 18, and 20, which involve either a cubic or a biquadratic equation, are applicable only for the case where x and y are expected to be integers. In fact, all the examples for the 21 rules in the text have integer solutions only. Rules 17 and 19 do not determine the solution uniquely.

Rule 1. $a, b \to x = (a + b)/2$, $y = (a - b)/2$.

Rule 2. $a, c \to b = \sqrt{a^2 - 4c}$ [→ Rule 1].

Rule 3. $a, d \to c = (a^2 - d)/2$ [→ Rule 2 → Rule 1].

Rule 4. $a, e \to b = e/a$ [→ Rule 1].

Rule 5. $a, f \to c = (a^3 - f)/3a$ [→ Rule 2 → Rule 1].

Rule 6. $a, g \to b$ is a quotient of division of $4g$ by $3a^2$ on the condition that the remainder equals the cube of that quotient: $4g = 3a^2b + b^3$ [→ Rule 1].

Rule 7. $b, c \to a = \sqrt{b^2 + 4c}$ [→ Rule 1].

Rule 8. $b, d \to c = (d - b^2)/2$ [→ Rule 7→ Rule 1].

Rule 8a. $b, d \to a = \sqrt{2d - b^2}$ [→ Rule 1].

Rule 9. $b, e \to a = e/b$ [→ Rule 1].

Rule 10. $b, f \to a$ is a cube root of $4f$ on the condition that the remainder equals that root times $3b^2$: $4f = a^3 + 3ab^2$ [→ Rule 1].

Rule 11. $b, g \to c = (g - b^3)/3b$ [→ Rule 7 → Rule 1].

Rule 12. $c, d \to a = \sqrt{d + 2c}$, $b = \sqrt{d - 2c}$ [→ Rule 1].

Rule 13. $c, e \to d = \sqrt{4c^2 + e^2}$ [→ Rule 16].

Rule 14. $c, f \to g = \sqrt{f^2 - 4c^3}$ [→ Rule 21].

Rule 15. $c, g \to b$ is a quotient of division of g by $3c$ on the condition that the remainder equals the cube of that quotient: $g = 3cb + b^3$ [→ Rule 7 → Rule 1].

Rule 16. $d, e \to x^2 = (d + e)/2$, $y^2 = (d - e)/2$.

Rule 17. $d, f \to y$ is a quotient of division of f by d on the condition that the remainder is divisible by $(d - y^2)$, and b is the other factor in the remainder: $f = dy + (d - y^2)b$ [→ $x = y + b$].

Rule 18. $d, g \to b$ is a quotient of division of $(2g + b^3)$ by $3d$ on the condition that the division leaves no remainder: $2g + b^3 = 3db$ [→ Rule 8a → Rule 1].

Rule 19. $e, f \to x$ is a quotient of division of f by e on the condition that the remainder is divisible by $(x^2 - e)$, and a is the other factor in the remainder: $f = ex + (x^2 - e)a$ [$\rightarrow y = a - x$].

Rule 20. $e, g \rightarrow b$ is a quotient of division of $(3e^2 + b^4)$ by $4g$ on the condition that the division leaves no remainder: $3e^2 + b^4 = 4gb$ [$\rightarrow$ Rule 9 $\rightarrow$ Rule 1].

Rule 21. $f, g \rightarrow x^3 = (f + g)/2$, $y^3 = (f - g)/2$.

2.2. Rules, Examples, and Proofs

Śaṅkara gives the following introduction in prose: "Here (in this regard)[3] it has been taught by a best *brāhmaṇa* named Citrabhānu, a leading scholar who knows the reasoning (*yukti*) of mathematics (*gaṇita*) and of the sphere (*gola*), that the calculation of two ⟨unknown⟩[4] quantities should be made in twenty-one ways, when any two among the sum, difference, product, square, cube, and their roots (? *tanmūla*) of those two quantities are known. Only its direction is written here by us according to his instruction" [8, 109].

This is followed by three *śloka* stanzas by, probably, Śaṅkara himself:

> The sum, difference, product, the sum of the squares, the difference of them (the squares), the sum of the cubes, and the difference of the cubes of two ⟨unknown⟩ quantities: the calculation ⟨method⟩ of the two quantities from ⟨any⟩ two among these seven ⟨values⟩ told has been handed down ⟨to us⟩ in twenty-one ways. The sum is combined with the difference, etc., and so also is the difference with the product, etc. Likewise, the others, namely, the product, etc., too are combined with their own successors. It is told in such a way that one can separate the desired two ⟨quantities⟩. [8, 109]

After these introductory remarks Śaṅkara gives the 21 rules (called *sūtra*) of Citrabhānu in sixteen and one-half stanzas (unnumbered); paraphrases them in prose; and gives examples (*udāharaṇa*) in verse, step-by-step solutions to them in prose, and proofs of the rules again in verse [8, 110–126]. The rules and the examples are given in various meters such as *āryā, indravaṃśā, indravajrā, upajāti, viyoginī*, and *śloka*, but the proofs exclusively in the *śloka* meter.

The proofs are of two kinds, algebraic and geometric: the text proves Rules 1, 2, 3, 4, 5, 7, 8, 9, 12, 13, 14, and 20 algebraically; Rules 10, 11, 15, 17, 18, and 19 geometrically; and Rule 6 in both ways. The algebraic proofs will be rendered by us in the modern symbolic form, while the geometric ones will be fully translated with geometric figures supplied by us. The original text does not have figures but, in the proofs, solid figures are referred to by such words as *kṣetra* ("a field" or a geometric figure in general), *kuṭṭima* ("a paved floor" or a rectangular parallelepiped as a base), *pīṭha* ("a seat" or a rectangular parallelepiped as a base), *upapīṭha* ("a near seat" or a sub-base), and *bhitti* ("a wall" or an upright rectangular parallelepiped).

The provenance of the examples and the proofs is not certain. According to the last sentence of Śaṅkara's prose introduction, the ideas underlying them are presumably due to Citrabhānu, but it is most probably Śaṅkara who composed those Sanskrit verses.

[3] A pair of parentheses is used for an explanation of the preceding word(s).

[4] A pair of angular brackets is used for word(s) supplied by us.

Rule 1 + Rule 16 + Rule 21.

yogabhedau dvayor yuktaviśliṣṭau dalitāv ubhau /
mahadalpau kramād rāśī vargayor ghanayoś ca tau // 1 //
The sum and the difference of the sum and the difference of the two ⟨unknown quantities⟩, when halved severally, are the larger and the smaller quantities in order. Those (sum and difference) of the two squares, and of the two cubes, too, ⟨when treated likewise, give rise to the two squares and the two cubes⟩.

$$x = \frac{a+b}{2},\ y = \frac{a-b}{2} \cdots \text{Rule 1}$$

$$x^2 = \frac{d+e}{2},\ y^2 = \frac{d-e}{2} \cdots \text{Rule 16}$$

$$x^3 = \frac{f+g}{2},\ y^3 = \frac{f-g}{2} \cdots \text{Rule 21}$$

Remark. According to the editor, the Malayalam manuscript has separate stanzas for Rules 16 and 21. See his footnotes [8, 121, 126], in the former of which he explains the omission of the stanza for Rule 16 as due to haplography. It is, however, very likely either that Śaṅkara omitted it intentionally or that the manuscript he used did not contain those stanzas, because what have been omitted are exactly those two stanzas that prescribe the two rules (16 and 21) mentioned already in Stanza 1.

Example. (Hereafter, numerals above arrows indicate the serial numbers of the rules employed). $a = 500,\ b = 110 \xrightarrow{1} x = 305,\ y = 195.$

Proof (in one stanza). $(x + y) + (x - y) = 2x,\ (x + y) - (x - y) = 2y.$

Rule 2.

rāśyor dvayor yogakṛteś caturguṇaṃ ghātaṃ tyajec chiṣṭapadaṃ tadantaram // 2ab //
One should subtract four times the product of the two quantities from the square of their sum. The square root of the remainder is their difference. (That is, $b = \sqrt{a^2 - 4c}$.)

Example. $a = 101,\ c = 2394 \xrightarrow{2} b = 25 \xrightarrow{1} x = 63,\ y = 38.$

Proof (in one stanza). $(x + y)^2 - 4xy = (x - y)^2.$

Rule 3.

ghātas tu rāśyor atha yogavargatas tadvargayoge rahite ’rdhite bhavet // 2cd //
The product, on the other hand, of the two quantities will be ⟨obtained⟩ when the sum of their squares is subtracted from the square of their sum, and ⟨the remainder is⟩ halved. (That is, $c = (a^2 - d)/2$.)

Example. $a = 101,\ d = 5413 \xrightarrow{3} c = 2394 \xrightarrow{2} b = 25 \xrightarrow{1} x = 63,\ y = 38.$

Proof (in one stanza). $x^2 + y^2 - 2xy = (x - y)^2 = (x + y)^2 - 4xy$ [hence $(x + y)^2 - (x^2 + y^2) = 2xy$].

Rule 4.

vargāntarād yogahṛtas tu bhedo rāśyos tataḥ saṅkramaṇāt pṛthak tau // 3ab //
⟨When⟩ the difference of the squares is divided by the sum, the difference of the two quantities ⟨will be obtained⟩. Thence those two ⟨quantities are to be⟩ separated by means of the rule of concurrence. (That is, $b = e/a$.)

Remark. See Section 3 of this paper for the rule of concurrence, and Section 2 of the Appendix for Śaṅkara's peculiar expression of the division.

Example. $a = 50, e = 400 \xrightarrow{4} b = 8 \xrightarrow{1} x = 29, y = 21$.

Proof (in one stanza). $(x + y)(x - y) = x^2 - y^2$; hence $(x^2 - y^2)/(x + y) = x - y$.

Rule 5.

yogaghanād ghanayoge tyakte triguṇena rāśiyogena /
śiṣṭe hṛte 'tha rāśyor dvayor bhaved iṣṭayor ghātaḥ // 4 //
When the sum of the cubes is subtracted from the cube of the sum, and when the remainder is divided by the sum of the ⟨two⟩ quantities as multiplied by three, then the product of the two desired quantities will be ⟨obtained⟩. (That is, $c = (a^3 - f)/3a$.)

Example. $a = 25, f = 4375 \xrightarrow{5} c = 150 \xrightarrow{2} b = 5 \xrightarrow{1} x = 15, y = 10$.

Proof (in one stanza). $x^3 + y^3 + 3(x + y)xy = (x + y)^3$.

Rule 6.

ghanāntarād vedaguṇāt trinighnyā yogasya kṛtyā vibhajet phalaṃ yat /
rāśyantaraṃ syād yadi tatra śiṣṭāc chodhyas tayor bhedaghano yadi syāt // 5 //
One should divide four times the difference between the cubes by three times the square of the sum. The quotient will be the difference of the ⟨two unknown⟩ quantities if and only if the cube of the difference of the two (i.e., the quotient) can be subtracted from the remainder of that (i.e., the division). (That is to say, $4g = 3a^2b + b^3$, where $4g$ is the dividend, $3a^2$ the divisor, b the quotient to be obtained, and b^3 the remainder, which should be the cube of the quotient.)

Remark. This rule, which does not always give a solution, is based on the identity, $4(x^3 - y^3) = 3(x + y)^2(x - y) + (x - y)^3$.

Example. $a = 25, g = 2375 \xrightarrow{6} b = 5 \xrightarrow{1} x = 15, y = 10$.

Remark. This example yields the quotient, 5, whose cube (125) happens to be equal to the remainder (125) of the division. In order to explain the condition indicated by the *"if"* clause, we cite the example given by Gupta [1, 5], which does not occur in the text.

Given $a = 11$, $g = 999$, then $3a^2 = 363$, and $4g = 3996$. We have $3996 = 363 \times 11 + 3$ $(3 < 11^3)$; $3996 = 363 \times 10 + 366$ $(366 < 10^3)$, and $3996 = 363 \times 9 + 729$ $(729 = 9^3)$. In the first two divisions, the cube of the quotient is greater than the remainder of the division, and therefore the former cannot be subtracted from the latter. In the third, however, the cube of the quotient is equal to the remainder,

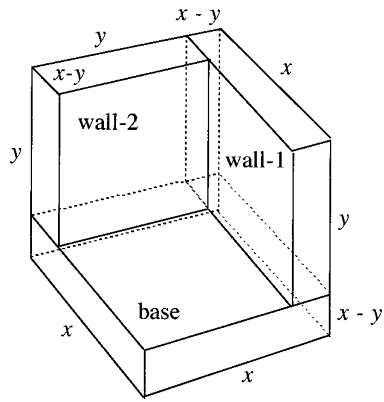

FIGURE 1

and the former "can be subtracted" from the latter. Hence $b = 9$, and from Rule 1, we have $x = 10$, $y = 1$.

Proof (in sixteen and one-half stanzas). The first four stanzas give an algebraic proof, and the rest (Stanzas 5–17ab) a geometric one. The algebraic proof, however, seems to deal with Rule 5 rather than Rule 6.

Algebraic proof (Stanzas 1–4). According to *Līlāvatī* 26, $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$. Hence, $(x + y)^3 - (x^3 + y^3) = 3xy(x + y)$. Therefore, $\{(x + y)^3 - (x^3 + y^3)\} \div 3(x + y) = xy$, so Rule 5 follows. It is possible to prove Rule 6 by substituting $-y$ for y in the first identity and adding $3(x^3 - y^3)$ to both sides. It is impossible to read this proof in the first three stanzas, and the next stanza (4) seems to state Rule 6 itself: "It is told in such a way that four times the difference of the cubes is larger than three times the square of the sum by the cube of the difference." An algebraic proof of the identity on which Rule 6 is based is given as the first step of the proof of Rule 20.

Geometric proof (Stanzas 5–17ab). See Figs. 1–3.

> One should cut two walls out of the base part of ⟨a figure representing⟩ the difference of the ⟨two⟩ cubes, ⟨a figure⟩ which stands like walls and a base closely joined together at two sides. One should know that the base below has four sides equal to the greater quantity (x) and a height equal to the difference of the ⟨two⟩ quantities ($x - y$) (the base in Fig. 1). Having made the two walls straight and laid ⟨them⟩ down, one should increase their length ⟨by doing so⟩. Then it will become ⟨an oblong⟩ having the sum of the two quantities ($x + y$) as length and the small quantity (y) as width (wall-1 + wall-2 in Fig. 2). To its side one should attach ⟨a square figure (i.e., the base)⟩ having the greater quantity as its four sides in such a way that it would become a figure having four sides equal to the sum ⟨of x and y⟩ except that it lacks ⟨an oblong⟩ whose width and length are the smaller and the greater quantities, respectively. With these three ⟨kinds of figures⟩ (the base, wall-1, and wall-2) joined together in this way, there will be triad of ⟨composed⟩ figures (Fig. 2).

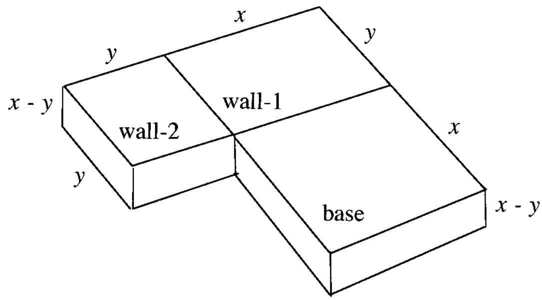

FIGURE 2

In that triad of figures ⟨each⟩ equal to the square of the sum, however, ⟨an oblong⟩ whose length and width are, respectively, the greater and the smaller quantities is wanting. One should make those three ⟨oblongs⟩. ⟨That is,⟩ having cut ⟨three pieces⟩ out of ⟨another⟩ figure for the difference of the cubes, one should fill it ⟨with them⟩. Now a method for it: One should cut one whole wall out of it (wall-1 in Fig. 3) because it (the wall) has a width and a length equal, respectively, to the smaller and the greater quantities. One should cut out another wall accompanied by ⟨a part of⟩ the base (wall-2 in Fig. 3). That, too, which has been cut out in this way, will be the same as the previous figure (wall-1). The remaining part of the base (the base in Fig. 3) also has a length and a width equal to them. ⟨A figure representing⟩ the cube of the difference ⟨and embedded⟩ in this base juts out of it at a corner.

When the three figures (wall-1, wall-2, and the base in Fig. 3) are linked in that way ⟨told above to the three incomplete squares (Fig. 2)⟩, there are three figures, ⟨each of which is⟩ equal to the square of the sum. Therefore, ⟨it should be known⟩, when three ⟨figures⟩ whose length and width are, respectively, the greater and the smaller ⟨quantities⟩ have been separated from the fourth ⟨figure representing the⟩ difference of the cubes, the cube of the difference remains. Therefore, when four times the difference of the cubes ($4g$) is divided by ⟨three times⟩ the

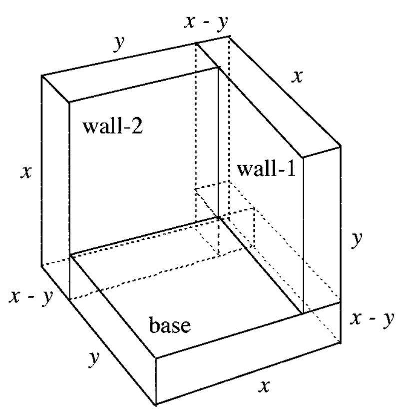

FIGURE 3

square of the sum ($3a^2$), since the cube of the difference (b^3) of the two quantities is left here (in the geometric consideration made above), it (b^3) is discarded. Since the thickness of those ⟨three square plates⟩ is equal to the difference (b) of the ⟨two⟩ quantities (in the geometric consideration made above), the fruit (the quotient of the division) is the difference (b).

Rule 7.

ghātāc caturguṇād bhedavargayuktāt padaṃ yutiḥ // 6ab //
The square root of the sum of four times the product and the square of the difference is the sum. (That is, $a = \sqrt{b^2 + 4c}$.)

Example. $b = 3, c = 108 \xrightarrow{7} a = 21 \xrightarrow{1} x = 12, y = 9$.

Proof (in one stanza). $(x + y)^2 = 4xy + (x - y)^2$.

Rule 8.

saṃvargo bhedavargonavargayogadalaṃ dvayoḥ // 6cd //
Half the sum of the squares of the two as decreased by the square of the difference is the product. (That is, $c = (d - b^2)/2$.)

Example. $b = 5, d = 625 \xrightarrow{8} c = 300 \xrightarrow{7} a = 35 \xrightarrow{1} x = 20, y = 15$.

Proof (in one stanza). $x^2 + y^2 = 2xy + (x - y)^2$.

Rule 9.

rāśyor bhedoddhṛto vargāntarād yogas tayor bhavet // 7ab //
The difference between the squares, when divided by the difference of the two quantities, will be the sum of the two. (That is, $a = e/b$.)

Remark. No example has been given.

Proof (in one stanza). $(x + y)(x - y) = x^2 - y^2$, so $(x^2 - y^2)/(x - y) = x + y$.

Rule 10.

ghanayogāc caturghnād yad ghanamūlaṃ tathākṛtam /
yogaḥ so 'ntaravargaghnas trighnas tyājyaś ca śeṣataḥ // 8 //
The cube root of four times the sum of the cubes is the sum ⟨of the two quantities, if it is⟩ made in such a way that that (i.e., the sum obtained), multiplied by the square of the difference and by three, can be subtracted from the remainder. (That is to say, $4f = a^3 + 3ab^2$, where a is the cube root to be extracted from $4f$, and $3ab^2$ the remainder, which should be the cube root multiplied by $3b^2$.)

Remark. This rule is based on the identity,

$$4(x^3 + y^3) = (x + y)^3 + 3(x + y)(x - y)^2.$$

Example. $b = 5, f = 11375 \xrightarrow{10} a = 35 \xrightarrow{1} x = 20, y = 15$.

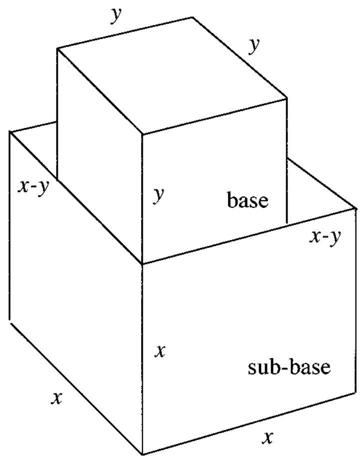

FIGURE 4

Proof (in twelve and one-half stanzas). See Figs. 4–7.

One should put down the cube figure of the smaller ⟨quantity⟩ on the cube of the greater in such a way that the two sides of ⟨each of⟩ the two ⟨cubes⟩ standing at a corner will be even (Fig. 4). Out[5] of the greater cube one should separate the two flanks located at the opposite corner, which jut out of the smaller cube. One should split the second ⟨figure representing the⟩ sum of the two cubes exactly in the same manner. ⟨Each of⟩ those two figures just separated has the form of two side walls; its width is equal to the greater ⟨quantity⟩, and its thickness to the difference of the ⟨two⟩ quantities.[6] One should put down what has been separated from the previous figure onto the sub-base of ⟨each of⟩ two other, non-cut, ⟨figures representing the⟩ sum of the cubes, where one having the form of a base is located above and one like a sub-base below (Fig. 5). What remains there has a length equal to the sum of the ⟨two⟩ quantities ($x + y$), and has four sides equal to the difference ($x - y$). Therefore, ⟨the L-shaped figure having been removed⟩, that will become ⟨a figure⟩ which has a height equal to the sum and four sides equal to the greater ⟨quantity⟩. The other two figures have a height equal to the sum and four sides equal to the smaller ⟨quantity⟩.

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[5] We have deleted Stanza 2ab which had appeared before this sentence, i.e., Stanza 2cd, since the latter seems to be a corrected version of the former, which does not make sense: *pratikoṇāśritaṃ pārśvadvayaṃ tu mahatā ghanam / pratikoṇāśritaṃ pārśvadvayaṃ mahato ghanāt // 2 //.*

[6] We have omitted, as in mss. A and B, Stanza 5 located after this sentence, since the stanza does not fit in this context. Stanza 5 reads: "After the coupling ⟨of the flank and the two cubes⟩, what extends from the inner, greater ⟨part⟩ is made straight. These two ⟨figures⟩, having the thickness equal to the difference of the ⟨two⟩ quantities, have a form equal to each other." We read here *antarmahatas* ("from the inner, greater") in the ablative case instead of *antarmahatā* (in the instrumental case) of the text. The first word, *ṛjūkṛtaṃ* ("made straight"), may refer to the process of "making straight" the two L-shaped figures cut out of the upper part in Fig. 5. In that case, the appropriate place for Stanza 5 would be after Stanza 7cd ("What remains there … equal to the difference").

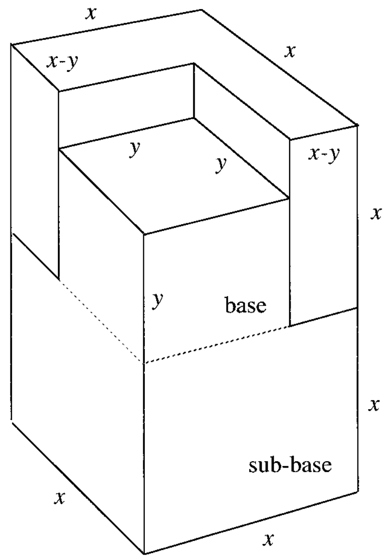

FIGURE 5

With these four ⟨figures⟩ joined together, there will be a figure whose thickness is the sum. ⟨It is as follows.⟩ Having removed a column-like ⟨figure⟩ which has a height equal to the sum and four sides equal to the difference from a corner of one of the greater ⟨figures⟩ (Fig. 6), one should join the two at their corners (Fig. 7). Those two smaller squares should be linked to the remaining two corners. Since those three ⟨column-like figures⟩, one of which has been removed from a corner (Fig. 6) and the other two from the upper part (Fig. 5), have a length equal to the sum and four sides equal to the difference, therefore the sum multiplied by the square of the difference and by three ($3ab^2$) is subtracted from four times the sum of the cubes ($4f$). When its cube root is extracted, the sum will be ⟨obtained⟩, when the sum multiplied by the square of the difference and by three is subtracted.[7]

Rule 11.

ghanāntarād dvayor bhedaghanonād yat samuddhṛtam /
trighnena rāśyor bhedena ghātaḥ sa tu bhavet tayoḥ // 9 //
When the difference between the cubes of the two ⟨quantities⟩ is decreased by the cube of the difference and divided by three times the difference, it (i.e., the quotient) will be the product of the two. (That is, $c = (g - b^3)/3b$.)

Remark. This rule is based on the identity, $x^3 - y^3 = (x - y)^3 + 3xy(x - y)$.

Example. $b = 5$, $g = 4625 \xrightarrow{11} c = 300 \xrightarrow{7} a = 35 \xrightarrow{1} x = 20$, $y = 15$.

[7] The phrase beginning with the second "when" in this sentence is superfluous.

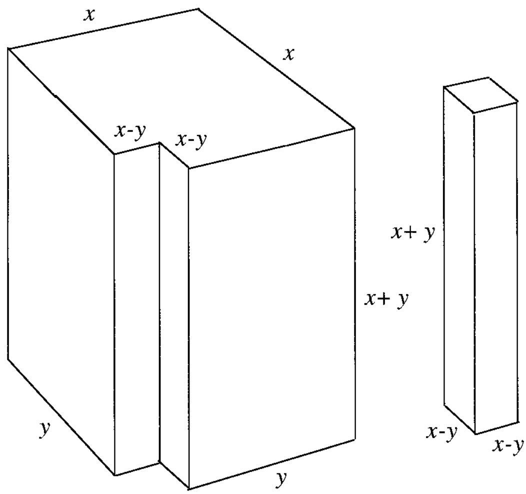

FIGURE 6

Proof (in four stanzas). See Fig. 3.

When two cubic figures ($(x - y)^3$ and y^3) are stuck to each other at their corners, three figures (wall-1, wall-2, and the base in Fig. 3), whose width and thickness are those two quantities ($x - y$ and y), and whose length is the sum (x) of the two, should be linked to each of three sides of the two ⟨cubes⟩ in such a way that the cube of the sum ⟨will be obtained⟩ by means of those five figures stuck together. When the cube of the smaller (y^3) is removed from the cube of the greater (x^3), the difference of the cubes ($x^3 - y^3$) remains. ⟨Again,⟩ when the cube of the difference ($(x - y)^3$) is removed from it (i.e., from the remainder), the two cubes have been separated ⟨from the large cube previously constructed⟩. There still remain three products ($3xy(x - y)$) of the three ⟨quantities⟩, that is, the two parts ($x - y$ and y) and the whole (x). So, the product of the two (xy) will be obtained from ⟨the division of⟩ it by three times the difference ($3(x - y)$).

Rule 12.

dvighnena ghātena yutonite yutī kṛtyoḥ pade dve khalu yogabhedau // 3cd //
The sum of the two squares is increased ⟨in one place⟩, and decreased ⟨in another⟩, by twice the product. The two square roots ⟨of the two results⟩ are indeed the sum and the difference. (That is, $a = \sqrt{d + 2c}$, $b = \sqrt{d - 2c}$.)

Example. $c = 300$, $d = 625 \xrightarrow{12} a = 35$, $b = 5 \xrightarrow{1} x = 20$, $y = 15$.

Proof (in one stanza). $x^2 + y^2 \pm 2xy = (x \pm y)^2$.

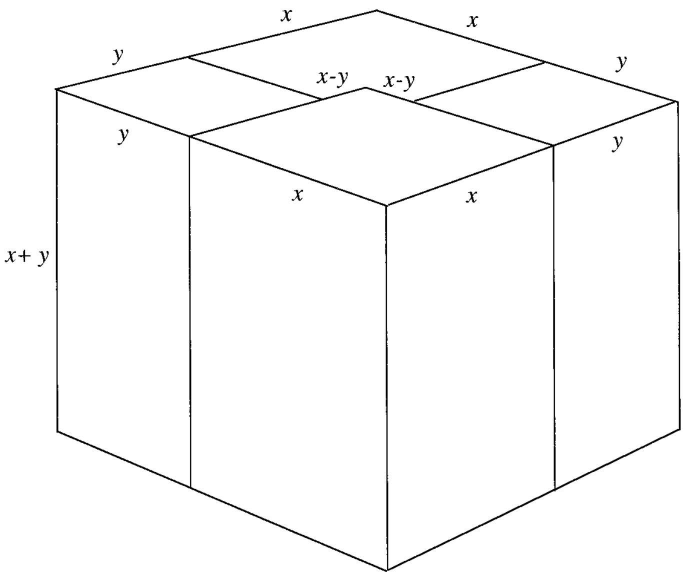

FIGURE 7

Rule 13 + Rule 16.

ghātasya vargād guṇitāc caturbhis tadvargabhedasya yutāc ca kṛtyā /
mūlaṃ bhaved vargayutis tu kṛtyor yogāntarābhyāṃ pṛthag eva vargau // 10 //
The square root of the sum of four times the square of the product and the square of the difference of their squares will be the sum of the squares. The two separate squares ⟨will be obtained⟩ from the sum and the difference of the two squares. (That is, $d = \sqrt{4c^2 + e^2}$ (Rule 13); $x^2 = (d + e)/2$, $y^2 = (d - e)/2$ (Rule 16).)

Example. $c = 300$, $e = 175 \xrightarrow{13} d = 625 \xrightarrow{16} x^2 = 400$, $y^2 = 225 \rightarrow x = 20$, $y = 15$.

Proof (in three stanzas). According to *Līlāvatī* 20, $(x - y)^2 + 4xy = (x + y)^2$. Substituting x^2 and y^2 for x and y in this identity, we have, $(x^2 - y^2)^2 + 4x^2y^2 = (x^2 + y^2)^2$. Hence, $(x^2 - y^2)^2 + 4(xy)^2 = (x^2 + y^2)^2$, so Rule 13 follows.

Rule 14.

caturguṇaṃ ghātaghanaṃ ghanayogakṛtes tyajet /
śeṣād yad vargamūlaṃ tad ghanayor antaraṃ bhavet // 11 //
One should subtract four times the cube of the product from the square of the sum of the cubes. The square root of the remainder will be the difference of the two cubes. (That is, $g = \sqrt{f^2 - 4c^3}$.)

Example. $c = 300, f = 1375 \xrightarrow{14} g = 4625 \xrightarrow{21} x^3 = 8000, y^3 = 3375 \rightarrow x = 20, y = 15$.

Proof (in four stanzas). According to *Līlāvatī* 20, $(x - y)^2 + 4xy = (x + y)^2$. Substituting x^3 and y^3 for x and y in this identity, we have, $(x^3 - y^3)^2 + 4x^3y^3 = (x^3 + y^3)^2$. Hence, $(x^3 - y^3)^2 = (x^3 + y^3)^2 - 4(xy)^3$. The proof is parallel to that of Rule 13.

Rule 15.

> *ghanabhedāt trighnaghātabhakto bhedo bhaved dvayoḥ /*
> *śiṣṭāl labdhaghane śuddhe prāgvad rāśidvayaṃ tataḥ // 12 //*
> The difference of the cubes, when divided by three times the product, will become the difference of the two ⟨quantities⟩, when the cube of the quotient is subtracted from the remainder. Thence the two quantities ⟨are obtained⟩ as before. (That is to say, $g = 3cb + b^3$, where g is the dividend, $3c$ the divisor, b the quotient to be obtained, and b^3 the remainder, which should be the cube of the quotient.)

Remark. Another rule is possible for the pair (c, g), $f = \sqrt{g^2 + 4c^3}$ (Rule 15a), which resembles Rule 14. In giving Rule 15 instead of Rule 15a, Citrabhānu has probably been influenced by Rule 11.

Example. $c = 300, g = 4625 \xrightarrow{15} b = 5$ [$\xrightarrow{7} a = 35 \xrightarrow{1} x = 20, y = 15$].

Proof (in four and one-half stanzas).

> It has been told ⟨in the proof of Rule 11⟩ that in the cube of a whole (x) there are ⟨five parts:⟩ the cubes of two parts (y and $x - y$) and three times the product of the two parts and the whole ($x^3 = y^3 + (x - y)^3 + 3xy(x - y)$).
>
> When one cube out of them is discarded, four parts remain; that is, the cube of the other part and three times the product of the two parts and the whole
>
> $$(x^3 - y^3 = (x - y)^3 + 3xy(x - y)).$$
>
> The difference of the cubes of the larger part (y) and the whole (x) is intended, and the product, too, of the same two. If, therefore, that difference of the cubes is divided by three times the product, there will be the difference ⟨for the quotient of the division⟩, and the remainder is its cube. When, therefore, the cube of the difference obtained ⟨as a quotient⟩ is ⟨exactly⟩ subtracted from the remainder of the division of the difference of the cubes by three times the product, it (the quotient) will be the true difference.

Remark. After giving this proof, the author prescribes the use of Rules 7 and 1 in the subsequent stanza.

Rule 16. See under Rules 1 and 13.

Rule 17.

> *ghanayogād vargayogavibhakto rāśir alpakaḥ /*
> *śiṣṭāc cel labdhavargonavargayogāptam antaram // 13 //*
> The sum of the cubes, when divided by the sum of the squares, ⟨gives rise to⟩ the smaller quantity, if the remainder, when divided by the sum of the squares decreased by the square of the quotient, ⟨gives rise to⟩ the difference. (That is, $f = dy + (d - y^2)b$, where f is the

dividend, d the divisor, y the quotient to be obtained, and $(d - y^2)b$ the remainder, which should be divisible by $(d - y^2)$.)

Remark. This rule, which is based on the identity, $x^3 + y^3 = (x^2 + y^2)y + x^2(x - y)$, does not determine the solution uniquely. When $d = 850$ and $f = 19{,}000$, for example, Rule 17 produces a wrong solution, $x = 24$ and $y = 10$, in addition to the correct one ($x = 25$, $y = 15$).

Example 1. $d = 100, f = 728 \xrightarrow{17} y = 6, b = 2 \rightarrow x = 8.$

Example 2. $d = 625, f = 11375 \xrightarrow{17} y = 15, b = 5 \rightarrow x = 20.$

Proof (in four and one-half stanzas).

From the cube of the greater ⟨quantity⟩, one should separate a figure whose height is the smaller quantity. That accompanied by the cube of the smaller ⟨quantity⟩, when divided by the sum of the squares, yields the smaller ⟨quantity⟩ ($(x^2y + y^3)/(x^2 + y^2) = y$). For exactly the same smaller ⟨quantity⟩ is obtained from ⟨the division of⟩ the cube of the smaller ⟨quantity⟩ by the square of the smaller ⟨quantity⟩, and likewise the smaller ⟨quantity⟩ is obtained from ⟨the division of⟩ the square of the greater ⟨quantity⟩ having the height equal to the smaller ⟨quantity⟩ by its square. Therefore, since the squares of the greater and the smaller ⟨quantities⟩ have a height equal to the smaller ⟨quantity⟩, the smaller quantity will be evidently obtained from ⟨the division of⟩ their sum by the sum of the squares. There remains a large square quadrilateral ($x^2(x - y)$) whose thickness is the difference ⟨of the two quantities⟩. The difference will be obtained from ⟨the division of⟩ that by the square of the greater ⟨quantity⟩. The square of the greater ⟨quantity⟩ in this case is the sum of the two squares decreased by the square of the smaller ⟨quantity⟩.

Rule 18.

dvighnād ghanāntarāl labhyaghanayuktād yad uddhṛtam /
trighnena vargayogena labdhaṃ syād antaraṃ dvayoḥ // 14 //
Twice the difference of the cubes increased by the cube of the quotient to be obtained is divided by three times the sum of the squares. The quotient will be the difference of the two ⟨quantities⟩. (That is to say, $2g + b^3 = 3db$, where $3d$ is the divisor, b the quotient to be obtained, and $2g + b^3$ the dividend.)

Remark. This rule is based on the identity,

$$2(x^3 - y^3) + (x - y)^3 = 3(x^2 + y^2)(x - y).$$

Example. $d = 625, g = 4625 \xrightarrow{18} b = 5 \xrightarrow{8a} a = 35 \xrightarrow{1} x = 20, y = 15.$

Remark. Note that, instead of Rule 8, the text employs another rule (8a). See the end of the proof.

Proof (in seven and one-half stanzas). See Fig. 1.

From ⟨a figure representing⟩ the difference of the ⟨two⟩ cubes which stands like walls and a base closely joined together at two sides, one should separate the base having a thickness equal to the difference; likewise ⟨separate⟩ another ⟨figure⟩ having a thickness equal to the difference and four sides equal to the smaller ⟨quantity⟩ (wall-2) from the wall; and likewise the third one which is the product of the greater and the smaller ⟨quantities⟩ (wall-1) ($2(x^3 - y^3) = 2x^2(x - y) + 2y^2(x - y) + 2xy(x - y)$).

From the combination of the first two among them there will be ⟨a figure representing⟩ the sum of the squares (the base and wall-2) having a thickness equal to the difference. The last one (wall-1) has a thickness equal to the difference and is measured by the product of the smaller and the greater ⟨quantities⟩. Now, one should suppose two other figures like them, and make another ⟨figure representing the⟩ sum of the squares by using these two ⟨figures representing the⟩ product. Since twice the product is less than the sum of the squares by the square of the difference, and since the thickness of all these figures is equal to the difference, therefore the "product figure" (the two wall-1's) in this case is less ⟨than the "square-sum figure"⟩ by the cube of the difference ($2xy(x - y) = (x^2 + y^2)(x - y) - (x - y)^3$).

Therefore, ⟨it has been said in Rule 18:⟩ twice the difference of the cubes increased by the quotient to be obtained is divided by three times the sum of the squares; the quotient will be the difference.

⟨Rule 8a:⟩ The square root of twice the sum of the squares decreased by the square of the difference will be the sum. One should obtain the two ⟨original⟩ quantities as before (Rule 1) from the sum and the difference. (That is, $a = \sqrt{2d - b^2}$.)

Rule 19.

ghanayogād vargabhedalabdho rāśir mahān iha /
śiṣṭāc ced vargabhedonalabdhavargoddhṛtā yutiḥ // 15 //

The quotient of ⟨division of⟩ the sum of the cubes by the difference of the squares is the greater quantity in this case, if the remainder, when divided by the square of the quotient decreased by the difference of the squares, ⟨gives rise to⟩ the sum. (That is to say, $f = ex + (x^2 - e)a$, where f is the dividend, e the divisor, x the quotient to be obtained, and $(x^2 - e)a$ the remainder, which should be divisible by $(x^2 - e)$.)

Remark. This rule, which is based on the identity, $x^3 + y^3 = (x^2 - y^2)x + y^2(x + y)$, is parallel to Rule 17, and the remark for the latter is valid here also.

Example. $e = 175, f = 11375 \xrightarrow{19} x = 20, a = 35 \rightarrow y = 15.$

Proof (in three and one-half stanzas). See Fig. 4.

When the cube ⟨figure⟩ of the smaller ⟨quantity⟩ is put down onto the cube ⟨figure⟩ of the greater ⟨quantity⟩, ⟨a figure⟩ having the form of the difference of the squares ⟨embedded⟩ in the cube of the greater juts out of the cube ⟨figure⟩ of the smaller. Since that part which juts out here has a thickness equal to the greater quantity, ⟨the surplus⟩ divided by the difference of the squares will be the greater ⟨quantity⟩. The other part has a height equal to the sum and four sides equal to the smaller ⟨quantity⟩. Therefore, the division of it by the square of the smaller will yield the sum of the two quantities ⟨as the quotient⟩. The smaller quantity will be ⟨obtained⟩ when the greater is subtracted from the sum of the ⟨two⟩ quantities.

Remark. The concluding half verse of the proof gives the relation: $y = a - x$.

Rule 20.

vargāntarasyātha kṛtiṃ trinighnīṃ harec caturghnena ghanāntareṇa /
bhājye kṣipel labdhakṛteś ca vargaṃ labdhaṃ tu rāśyantaram āmananti // 16 //

One should divide three times the square of the difference of the squares by four times the difference of the cubes, and add the square of the square of the ⟨provisional⟩ quotient to the dividend ⟨so that it may be divisible by the divisor⟩. They consider the quotient to be the difference of the ⟨two⟩ quantities. (That is to say, $3e^2 + b^4 = 4gb$, where $3e^2 + b^4$ is the dividend, $4g$ the divisor, and b the quotient to be obtained.)

Remark. This rule is based on the identity,

$$3(x^2 - y^2)^2 + (x - y)^4 = 4(x^3 - y^3)(x - y).$$

Example. $e = 175,\ g = 4625 \xrightarrow{20} b = 5\ [\xrightarrow{9} a = 35 \xrightarrow{1} x = 20,\ y = 15]$.

Proof (in twenty-two stanzas). This is an algebraic proof consisting of two steps.

First step (Stanzas 1–9ab): Proof of $4(x^3 - y^3) = 3(x + y)^2(x - y) + (x - y)^3$. The text proceeds as follows.

$$\begin{aligned} x^3 = \{(x - y) + y\}^3 &= (x - y)^3 + 3(x - y)^2y + 3(x - y)y^2 + y^3 \\ &= (x - y)\{(x - y)^2 + 3(x - y)y + 3y^2\} + y^3 \\ &= (x - y)\{(x - y)^2 + 3xy\} + y^3. \end{aligned}$$

Hence,

$$x^3 - y^3 = 3xy(x - y) + (x - y)^3. \tag{1}$$

On the other hand, $(x + y)^2 = 4xy + (x - y)^2$. Thus, $(x + y)^2(x - y) = 4xy\ (x - y) + (x - y)^3$. Hence,

$$3(x + y)^2(x - y) = 12xy(x - y) + 3(x - y)^3. \tag{2}$$

From (1),

$$4(x^3 - y^3) = 12xy(x - y) + 4(x - y)^3. \tag{3}$$

By comparing (2) and (3),

$$4(x^3 - y^3) = 3(x + y)^2(x - y) + (x - y)^3. \tag{4}$$

This identity is proved geometrically in Rule 6.

Second step (Stanzas 9cd–22): Proof of Rule 20. "When one has established this in his mind, the reasoning (*yukti*) on the main point is explained ⟨as follows⟩."

$$\frac{3(x^2 - y^2)^2}{4(x^3 - y^3)} = \frac{3\{(x + y)(x - y)\}^2}{3(x + y)^2(x - y) + (x - y)^3} \approx \frac{3\{(x + y)(x - y)\}^2}{3(x + y)^2(x - y)} = x - y,$$

since[8] $\{(x + y)(x - y)\}^2 = (x + y)^2(x - y) \cdot (x - y)$. More precisely,

$$\frac{3\{(x + y)(x - y)\}^2}{4(x^3 - y^3)} < \frac{3\{(x + y)(x - y)\}^2}{3(x + y)^2(x - y)} = x - y,$$

by (4). By (4) also,

[8] In the five stanzas (15–19) that follow this identity, the author seems to try to explain it in general terms, but the details are not clear.

$$\frac{3(x^2 - y^2)^2}{4(x^3 - y^3) - (x - y)^3} = \frac{3(x^2 - y^2)^2}{3(x + y)^2(x - y)} = x - y.$$

Thus $3(x^2 - y^2)^2 = 4(x^3 - y^3)(x - y) - (x - y)^4$. Hence, $\{3(x^2 - y^2)^2 + (x - y)^4\} \div 4(x^3 - y^3) = x - y$. Rule 20 now follows.

Rule 21. See Rule 1.

3. NORMAL FORMS BEFORE CITRABHĀNU

We assume here the same notation as in the previous section. For the abbreviations of the titles of the Sanskrit works, see the References.

In A.D. 499 or a little later Āryabhaṭa gave the following two rules:

AB 2.23: $c = (a^2 - d)/2$ (Rule 3).

AB 2.24: $x = (\sqrt{b^2 + 4c} + b)/2$, $y = (\sqrt{b^2 + 4c} - b)/2$ (Rule 7 and Rule 1).

While commenting on the AB in A.D. 629, Bhāskara I [BAB,57,103] called Rule 1 *saṅkaramaṇa* (‘‘concurrence’’).

In A.D. 628 Brahmagupta gave the following four sets of formulas:

BSS 18.36ab and 96: $x = (a + b)/2$, $y = (a - b)/2$ (Rule 1).

BSS 18.36cd and 97: $x = (e/b + b)/2$, $y = (e/b - b)/2$ (Rule 9 and Rule 1).

BSS 18.98: $x = (a + \sqrt{2d - a^2})/2$, $y = (a - \sqrt{2d - a^2})/2$ (Rule 3a and Rule 1), where Rule 3a is $b = \sqrt{2d - a^2}$.

BSS 18.99: $x = (\sqrt{b^2 + 4c} + b)/2$, $y = (\sqrt{b^2 + 4c} - b)/2$ (Rule 7 and Rule 1).

The first two rules of Brahmagupta were repeated in about A.D. 850 by Mahāvīra [GSS 6.2], who called them respectively *saṅkramaṇa* and *viṣama-saṅkramaṇa* (‘‘odd concurrence’’); in about A.D. 950 (or 1500?) by Āryabhaṭa II [MS 15.21cd–22], who called the former *saṅkrama/saṅkramaṇa* while giving the condition, $x \neq y$ (*viṣamajātīyau),* for the latter; in about A.D. 1040 by Śrīpati [SS 14.13], who called them *saṅkramaṇa-karman* and *viṣama-karman,* respectively; and before A.D. 1150 by Bhāskara II [L 56,58], who called them *saṅkramaṇa* and *varga-saṅkramaṇa*, respectively, and the two rules *viṣamakarman* collectively. Śaṅkara follows Bhāskara II’s terminolgy (see Rule 4 above). Mahāvīra and Bhaškara II gave the following examples.

GSS 6.3: $a = 12, b = 2 \xrightarrow{1} x = 7, y = 5$; $b = 2, e = 12 \xrightarrow{9} a = 6 \xrightarrow{1} x = 4, y = 2$.

L 57 and 59: $a = 101, b = 25 \xrightarrow{1} x = 63, y = 38$; $b = 8, e = 400 \xrightarrow{9} a = 50 \xrightarrow{1} x = 29, y = 21$.

In A.D. 1356 Nārāyaṇa gave the following eleven rules (Stanzas 31–37ab) accompanied by examples in a chapter called *prakīrṇaka* (‘‘miscellaneous problems’’) of his *Gaṇitakaumudī.*

Stanza 31: $x = (a + b)/2$, $y = (a - b)/2$ (Rule 1); Ex. 13: $a = 63, b = 9 \xrightarrow{1} x = 36, y = 27$.

Stanza 32: $a = e/b$ (Rule 9), $b = e/a$ (Rule 4); Ex. 14: $b = 8, e = 400 \xrightarrow{9} a = 50 \xrightarrow{1} x = 29, y = 21$; $a = 100, e = 400 \xrightarrow{4} b = 4 \xrightarrow{1} x = 52, y = 48$.

Stanza 33: $a = \sqrt{2d - b^2}$ (Rule 8a); Ex.: $b = 2$, $d = 100 \xrightarrow{8a} a = 14 \xrightarrow{1} x = 8$, $y = 6$.

Stanza 34: $d = \sqrt{4c^2 + e^2}$ (Rule 13), $x^2 = (d + e)/2$, $y^2 = (d - e)/2$ (Rule 16); Ex.: $c = 300$, $e = 175 \xrightarrow{13} d = 625 \xrightarrow{16} x^2 = 400$, $y^2 = 225 \rightarrow x = 20$, $y = 15$.

Stanza 35: $a = \sqrt{b^2 + 4c}$ (Rule 7), $b = \sqrt{a^2 - 4c}$ (Rule 4a); Ex. 15: $b = 7$, $c = 60 \xrightarrow{7} a = 17 \xrightarrow{1} x = 12$, $y = 5$.

Stanza 36: $d = b^2 + 2c$ (Rule 7a), $a = \sqrt{2c + d}$ (Rule 12); Ex. 16: $b = 5$, $c = 300 \xrightarrow{7a} d = 625 \xrightarrow{12} a = 35 \xrightarrow{1} x = 20$, $y = 15$.

Stanza 37ab: $b = \sqrt{2d - a^2}$ (Rule 3a); Ex. 17: $a = 14$, $d = 100 \xrightarrow{3a} b = 2 \xrightarrow{1} x = 8$, $y = 6$.

Nārāyaṇa gives three synonymous names, *saṅkramaṇa, saṅkrama*, and *saṅkrāma*, for Rule 1 (GK prakīrṇaka 31). The commentator[9] introduces Stanzas 32, 33, and 34 by saying, "a rule for another *saṅkramaṇa*" (*saṅkramaṇāntare sūtram*). Moreover, this section ends with the words, "thus ⟨ends⟩ *saṅkramaṇa*." It is, therefore, likely that all the rules for the algebraic normal forms were called *saṅkramaṇa* at least by the commentator, if not by Nārāyaṇa himself. Before the ending remark we read that "the rest is useful for geometry. I will explain ⟨it⟩ there (in the section of geometry)." We, however, have not so far been able to identify the place.

In conclusion, Table 1 gives the locations of each rule, where C–S indicates "Citrabhānu as handed down to us by Śaṅkara."

APPENDIX

1. Corrections made in the text.

Pages	Lines	Corrections
114	1	*svalpaṃ mahad-* → *svalpamahad-* (as in mass. A and B).
	13	*tritayaṃ* → *tritaye* (cf. *titaye* in mss. A and B).
116	20	Stanza 2ab → Delete.
118	18	*yogaghanakṣetraiḥ* → *yogaghanaṃ kṣetraiḥ.*
	22	*syādyutis* → *syāddhatis.*
119	24	*yogavarga-* → *vargayoga-.*
	25	*mūlene* → *mūlena.*
122	8	*-yogodghṛtas* → *-yogoddhṛtas.*
123	8	*saṃvarge* → *saṃvargaṃ* (as in mss. C and D).
124	2	*'lpasya ca* → *vargasya ca* (as in mss. A and B).
	13	*-raṃ kṛtiḥ* → *-rākṛti* (cf. *-rākṛtiḥ* in mss. A, B, C, D).
125	23	*bhidā ghanaḥ* → *bhidāghanaḥ.*
126	1	*bhidā kṛtau* → *bhidākṛtau.*
	13	*vathaḥ* → *vadhaḥ.*
	19	*cirāhatiḥ* → *dvirāhatiḥ.*

[9] It is not certain whether the commentator was Nārāyaṇa himself. See [2, 199–202].

TABLE 1
Locations of Rules for Normal Forms

Rule	AB	BSS	GSS	MS	SS	L	GK	C–S
1	◦	◦	◦	◦	◦	◦	◦	◦
2								◦
3	◦							◦
3a		◦					◦	
4							◦	◦
4a							◦	
5								◦
6								◦
7	◦	◦					◦	◦
7a							◦	
8								◦
8a							◦	◦
9		◦	◦	◦	◦	◦	◦	◦
10								◦
11								◦
12							◦	◦
13							◦	◦
14								◦
15								◦
16							◦	◦
17								◦
18								◦
19								◦
20								◦
21								◦

2. Śaṅkara's peculiar expression of the division in Rules 4, etc.

In Rule 4 and several other rules, Śaṅkara's expression of the division shows an anomaly from the regular Sanskrit construction. He puts the past passive participle (*hṛta)* of the verb ($\sqrt{}$*hṛ*, "to take") in the nominative case while the dividend (*vargāntara*) in the ablative, and makes the gender of the past participle agree with that of the quotient (*bheda*): dividend (abl.) + pp. of verb (nom.) + quotient (nom.). The divisor is either made into a compound with the past participle or put in the instrumental case. See Rules 9, 11, 15, 17 (twice), 18, and 19. In Rule 6 he still puts the dividend in the ablative case while the verb takes the optative form: dividend (abl.) + divisor (instr.) + opt. of verb.

3. "If" clause in Rule 6.

The repetition of the word *yadi* ("if") in Stanza 5 seems to mean "if and only if," although such a usage has not been attested elsewhere.

4. Meter of Rule 12.

The half stanza for Rule 12, the meter of which is irregular (a quarter *indravaṃśā* + a quarter *indravajrā*), seems to be coupled with that for Rule 4, a half *indravajrā*.

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Abbreviations of the Sanskrit Texts

AB: *Āryabhaṭīya* of Āryabhaṭa I, edited with the commentaries of Bhāskara I and of Someśvara by K. S. Shukla, New Delhi, 1976.

BAB: Bhāskara I's commentary on the AB. See AB above.

BSS: *Brāhmasphuṭasiddhānta* of Brahmagupta, edited by S. Dvivedī, Benares, 1902.

GK: *Gaṇitakaumudī* of Nārāyaṇa, edited by P. Dvivedī, Princess of Wales Sarasvati Bhavana Texts 57, Benares, 1936/42. Chapters 13 and 14 have been edited with an English translation by Takanori Kusuba. See [2] above.

GSS: *Gaṇitasārasaṃgraha* of Mahāvīra, edited with an English translation by M. Raṅgācārya, Madras, 1912. Also edited with a Hindī translation by L. C. Jain, Sholapur, 1963.

GT: *Gaṇitatilaka* of Śrīpati, edited with the commentary of Siṃhatilaka Sūri by H. R. Kāpadīā, Gaekwad's Oriental Series 78, Baroda, 1937.

KK: *Kriyākramakarī* of Śaṅkara and Nārāyaṇa, edited by K. V. Sarma. See [8] above.

L: *Līlāvatī* of Bhāskara II, edited with the commentaries of Gaṇeśa I and of Mahīdhara by V. G. Āpaṭe. Also edited by K. V. Sarma. See [8] above.

MS: *Mahāsiddhānta* of Āryabhaṭa II, edited with his own commentary by S. Dvivedī, Benares Sanskrit Series 148, 149, and 150, Benares, 1910. Reprinted, Delhi, 1995.

SS: *Siddhāntaśekhara* of Śrīpati, edited with the commentaries of Makkibhaṭṭa and of his own by B. Miśra, Calcutta, 1932/47.